

EXPLICIT AND UNIQUE CONSTRUCTION OF TETRABLOCK UNITARY DILATION IN A CERTAIN CASE

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ABSTRACT. Consider the domain E in \mathbb{C}^3 defined by

$$E = \{(a_{11}, a_{22}, \det A) : A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ with } \|A\| < 1\}.$$

This is called the tetrablock. This paper constructs explicit boundary normal dilation for a triple (A, B, P) of commuting bounded operators which has \overline{E} as a spectral set.

We show that the dilation is minimal and unique under a certain natural condition. As is well-known, uniqueness of minimal dilation usually does not hold good in several variables, e.g., Ando's dilation is not known to be unique. However, in the case of the tetrablock, the third component of the dilation can be chosen in such a way as to ensure uniqueness.

1. INTRODUCTION

Let K be a compact subset of \mathbb{C}^d for $d \geq 1$. Consider a d -tuple $\underline{T} = (T_1, T_2, \dots, T_d)$ of commuting bounded operators with K as a spectral set, i.e., the joint spectrum of \underline{T} is contained in K and

$$\|f(\underline{T})\| \leq \sup\{|f(z)| : z \in K\}$$

for all rational functions f with poles off K . A commuting tuple of bounded normal operators $\underline{N} = (N_1, N_2, \dots, N_d)$ with $\sigma(\underline{N}) \subset bK$, the distinguished boundary of K is called a *normal boundary dilation* of \underline{T} if

$$f(\underline{T}) = P_{\mathcal{H}} f(\underline{N})|_{\mathcal{H}},$$

for all rational functions f with poles off K . The K we consider in this paper, is a polynomially convex domain. Note that by Oka-Weil theorem, for a polynomially convex domain, a polynomial dilation is the same as a rational dilation. In other words,

$$T_1^{k_1} \dots T_d^{k_d} = P_{\mathcal{H}} N_1^{k_1} \dots N_d^{k_d}|_{\mathcal{H}}$$

for $k_1, \dots, k_d \geq 0$.

Definition 1. A triple (A, B, P) of commuting bounded operators on a Hilbert space \mathcal{H} is called a *tetrablock contraction* if \overline{E} is a spectral set for (A, B, P) , i.e., the joint spectrum of (A, B, P) is contained in \overline{E} and

$$\|f(A, B, P)\| \leq \|f\|_{\infty, \overline{E}} = \sup\{|f(x_1, x_2, x_3)| : (x_1, x_2, x_3) \in \overline{E}\}$$

for any polynomial f in three variables.

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Consider a tetrablock contraction (A, B, P) . Then it is easy to see that P is a contraction. Fundamental equations for a tetrablock contraction are introduced in [2]. These are

$$(1.1) \quad A - B^*P = D_P F_1 D_P, \text{ and } B - A^*P = D_P F_2 D_P$$

where $D_P = (I - P^*P)^{\frac{1}{2}}$ is the defect operator of the contraction P and $\mathcal{D}_P = \overline{\text{Ran}} D_P$ and F_1, F_2 are bounded operators on \mathcal{D}_P . Theorem 3.5 in [2] says that the two fundamental equations can be solved and the solutions F_1 and F_2 are unique. The unique solutions F_1 and F_2 of equations (1.1) are called the *fundamental operators* of the tetrablock contraction (A, B, P) . Moreover, $w(F_1)$ and $w(F_2)$ are not greater than 1. The adjoint triple (A^*, B^*, P^*) is also a tetrablock contraction as can be seen from definition. By what we stated above there are unique $G_1, G_2 \in \mathcal{B}(\mathcal{D}_{P^*})$ such that

$$(1.2) \quad A^* - B P^* = D_{P^*} G_1 D_{P^*} \text{ and } B^* - A P^* = D_{P^*} G_2 D_{P^*},$$

and $w(G_1)$ and $w(G_2)$ are not greater than 1.

Definition 2. A *tetrablock unitary* is a triple of commuting bounded operators $\underline{N} = (N_1, N_2, N_3)$ on a Hilbert space \mathcal{H} such that its Taylor joint spectrum $\sigma(\underline{N})$ is contained in bE , the Shilov boundary of E .

Definition 3. A *tetrablock isometry* is the restriction of a tetrablock unitary to a joint invariant subspace.

Let (A, B, P) be a tetrablock contraction on \mathcal{H} with fundamental operators F_1 and F_2 . Consider the Hilbert space $\tilde{\mathcal{H}} = \mathcal{H} \oplus l^2(\mathcal{D}_P)$. Let V_1, V_2 and V_3 be defined on $\tilde{\mathcal{H}}$ by

$$V_1(h \oplus (a_0, a_1, a_2, \dots)) = (Ah \oplus (F_2^* D_P h + F_1 a_0, F_2^* a_0 + F_1 a_1, F_2^* a_1 + F_1 a_2, \dots))$$

$$V_2(h \oplus (a_0, a_1, a_2, \dots)) = (Bh \oplus (F_1^* D_P h + F_2 a_0, F_1^* a_0 + F_2 a_1, F_1^* a_1 + F_2 a_2, \dots))$$

and

$$V_3(h \oplus (a_0, a_1, a_2, \dots)) = (Ph \oplus (D_P h, a_0, a_1, a_2, \dots))$$

respectively. From Theorem 6.1 of [2], we learnt that (V_1, V_2, V_3) on $\tilde{\mathcal{H}}$ is a tetrablock isometric dilation of (A, B, P) if F_1, F_2 satisfy

$$(1.3) \quad [F_1, F_2] = 0 \text{ and } [F_1, F_1^*] = [F_2, F_2^*].$$

Our first major result, described in the theorem below, is the construction of tetrablock unitary dilation of a tetrablock contraction explicitly.

Theorem 4. Let (A, B, P) be a tetrablock contraction on \mathcal{H} with fundamental operators F_1 and F_2 satisfying (1.3). Let the space $\tilde{\mathcal{H}}$ and the operator triple (V_1, V_2, V_3) be as above. Consider the space $\mathcal{K} = \tilde{\mathcal{H}} \oplus l^2(\mathcal{D}_{P^*})$. Define operators $C_1, C_2, C_3 : l^2(\mathcal{D}_{P^*}) \rightarrow \tilde{\mathcal{H}}$ by

$$\begin{aligned} C_1(a_0, a_1, a_2, \dots) &= (D_{P^*} G_2 a_0 \oplus (-F_2^* P^* a_0, 0, 0, \dots)), \\ C_2(a_0, a_1, a_2, \dots) &= (D_{P^*} G_1 a_0 \oplus (-F_1^* P^* a_0, 0, 0, \dots)), \\ C_3(a_0, a_1, a_2, \dots) &= (D_{P^*} a_0 \oplus (-P^* a_0, 0, 0, \dots)) \end{aligned}$$

and $D_1, D_2, D_3 : l^2(\mathcal{D}_{P^*}) \rightarrow l^2(\mathcal{D}_{P^*})$ by

$$\begin{aligned} D_1(a_0, a_1, a_2, \dots) &= (G_1^*a_0 + G_2a_1, G_1^*a_1 + G_2a_2, G_1^*a_2 + G_2a_3, \dots), \\ D_2(a_0, a_1, a_2, \dots) &= (G_2^*a_0 + G_1a_1, G_2^*a_1 + G_1a_2, G_2^*a_2 + G_1a_3, \dots) \text{ and} \\ D_3(a_0, a_1, a_2, \dots) &= (a_1, a_2, a_3, \dots) \text{ respectively.} \end{aligned}$$

Finally, let R_1, R_2 and U be three operators defined on \mathcal{K} whose block operator matrices with respect to the decomposition $\mathcal{K} = \tilde{\mathcal{H}} \oplus l^2(\mathcal{D}_{P^*})$ are

$$\begin{pmatrix} V_1 & C_1 \\ 0 & D_1 \end{pmatrix}, \begin{pmatrix} V_2 & C_2 \\ 0 & D_2 \end{pmatrix} \text{ and } \begin{pmatrix} V_3 & C_3 \\ 0 & D_3 \end{pmatrix} \text{ respectively.}$$

Then the triple (R_1, R_2, U) is a tetrablock unitary dilation of (A, B, P) .

This dilation is proved to be minimal. Unlike in the case of a single operator, minimality of a dilation of an n -tuple ($n > 1$) of commuting operators does not guarantee its uniqueness. We show that the tetrablock unitary dilation (R_1, R_2, U) defined in Theorem 4 of a tetrablock contraction (A, B, P) is unique under a certain suitable condition. The uniqueness theorem, the second major result of this paper, states that if $(\tilde{R}_1, \tilde{R}_2, \tilde{U})$ is a tetrablock unitary dilation of a tetrablock contraction (A, B, P) such that \tilde{U} is the minimal unitary dilation of P , then the dilation $(\tilde{R}_1, \tilde{R}_2, \tilde{U})$ is unitarily equivalent to the dilation we have constructed. This is the content of Theorem 14. Note that if a tetrablock unitary $(\tilde{R}_1, \tilde{R}_2, \tilde{U})$ dilates a tetrablock contraction (A, B, P) , then by definition of dilation, the unitary \tilde{U} dilates the contraction P too. So the only constraint we are imposing is that the dilation \tilde{U} is minimal.

Two equations associated with a contraction P and its defect operators that will come handy are

$$(1.4) \quad PD_P = D_{P^*}P$$

and its corresponding adjoint relation

$$(1.5) \quad D_P P^* = P^* D_{P^*}.$$

Proof of (1.4) and (1.5) can be found in [9](ch. 1, sec. 3). We shall use these two relations in this paper without mention.

Note that for a Hilbert space \mathcal{E} , the Hilbert space $l^2(\mathcal{E})$ is unitarily equivalent to the Hilbert space $H_{\mathcal{E}}^2(\mathbb{D})$, via the unitary map

$$(\xi_0, \xi_1, \xi_2, \dots) \mapsto \sum_{n=0}^{\infty} z^n \xi_n,$$

where $\xi_n \in \mathcal{E}$ for all $n \geq 0$.

2. ELEMENTARY RESULTS ON A TETRABLOCK CONTRACTION

All known facts about tetrablock contractions, tetrablock isometries and tetrablock unitaries that we quote here are from [2].

There are well known characterizations of a tetrablock isometry and a tetrablock unitary. We just quote some characterizations of a tetrablock unitary because we shall use it later in this paper.

Theorem 5. *Let $\underline{N} = (N_1, N_2, N_3)$ be a commuting triple of bounded operators. Then the following are equivalent:*

- (1) \underline{N} is a tetrablock unitary;
- (2) N_3 is a unitary, N_2 is a contraction and $N_1 = N_2^* N_3$;
- (3) there is a 2×2 unitary block operator matrix $[U_{ij}]$ where U_{ij} are commuting normal operators and $\underline{N} = (U_{11}, U_{22}, U_{11}U_{22} - U_{21}U_{12})$;
- (4) N_3 is a unitary and \underline{N} is a tetrablock contraction;
- (5) the family $\{(R_z, U_z) : |z| = 1\}$ where $R_z = N_1 + zN_2$ and $U_z = zN_3$ is a commuting family of Γ -unitaries.

For a proof, see [2](Th. 5.4). Before going to construct the tetrablock unitary dilation of a tetrablock contraction, let us study few lemmas which will be used in the construction. First we state a very important result from [2](Corollary 4.2).

Lemma 6. *The fundamental operators F_1 and F_2 of a tetrablock contraction (A, B, P) are the unique bounded linear operators on \mathcal{D}_P that satisfy the pair of operator equations*

$$D_P A = F_1 D_P + F_2^* D_P P \text{ and } D_P B = F_2 D_P + F_1^* D_P P.$$

The next three lemmas give relations between the fundamental operators of a tetrablock contraction and its adjoint.

Lemma 7. *Let (A, B, P) be a tetrablock contraction on a Hilbert space \mathcal{H} and F_1, F_2 and G_1, G_2 be fundamental operators of (A, B, P) and (A^*, B^*, P^*) respectively. Then*

$$P F_i = G_i^* P|_{\mathcal{D}_P}, \text{ for } i=1 \text{ and } 2.$$

Proof. We shall prove only for $i = 1$, the proof for $i = 2$ is similar. Note that the operators on both sides are from \mathcal{D}_P to \mathcal{D}_{P^*} . Let $h, h' \in \mathcal{H}$ be any element. Then

$$\begin{aligned} & \langle (P F_1 - G_1^* P) D_P h, D_{P^*} h' \rangle \\ &= \langle D_{P^*} P F_1 D_P h, h' \rangle - \langle D_{P^*} G_1^* P D_P h, h' \rangle \\ &= \langle P (D_P F_1 D_P) h, h' \rangle - \langle (D_{P^*} G_1^* D_{P^*}) P h, h' \rangle \\ &= \langle P (A - B^* P) h, h' \rangle - \langle (A - P B^*) P h, h' \rangle \\ &= \langle (P A - P B^* P - A P + P B^* P) h, h' \rangle = 0. \end{aligned}$$

Hence the proof. ■

Lemma 8. *Let (A, B, P) be a tetrablock contraction on a Hilbert space \mathcal{H} and F_1, F_2 and G_1, G_2 be fundamental operators of (A, B, P) and (A^*, B^*, P^*) respectively. Then*

$$D_P F_1 = (A D_P - D_{P^*} G_2 P)|_{\mathcal{D}_P} \text{ and } D_P F_2 = (B D_P - D_{P^*} G_1 P)|_{\mathcal{D}_P}.$$

Proof. We shall prove only one of the above, proof of the other is similar. For $h \in \mathcal{H}$, we have

$$\begin{aligned} (A D_P - D_{P^*} G_2 P) D_P h &= A(I - P^* P)h - (D_{P^*} G_2 D_{P^*}) P h \\ &= A h - A P^* P h - (B^* - A P^*) P h \\ &= A h - A P^* P h - B^* P h + A P^* P h \\ &= (A - B^* P) h = (D_P F_1) D_P h. \end{aligned}$$

Hence the proof. ■

Lemma 9. Let (A, B, P) be a tetrablock contraction on a Hilbert space \mathcal{H} and F_1, F_2 and G_1, G_2 be fundamental operators of (A, B, P) and (A^*, B^*, P^*) respectively. Then

$$\begin{aligned} (F_1^* D_P D_{P^*} - F_2 P^*)|_{\mathcal{D}_{P^*}} &= D_P D_{P^*} G_1 - P^* G_2^* \text{ and} \\ (F_2^* D_P D_{P^*} - F_1 P^*)|_{\mathcal{D}_{P^*}} &= D_P D_{P^*} G_2 - P^* G_1^*. \end{aligned}$$

Proof. For $h \in \mathcal{H}$, we have

$$\begin{aligned} (F_1^* D_P D_{P^*} - F_2 P^*) D_{P^*} h &= F_1^* D_P (I - P P^*) h - F_2 P^* D_{P^*} h \\ &= F_1^* D_P h - F_1^* D_P P P^* h - F_2 D_P P^* h \\ &= F_1^* D_P h - (F_1^* D_P P + F_2 D_P) P^* h \\ &= F_1^* D_P h - D_P B P^* h \quad [\text{by Lemma 6}] \\ &= (A D_P - D_{P^*} G_2 P)^* h - D_P B P^* h \quad [\text{by Lemma 8}] \\ &= D_P A^* h - P^* G_2^* D_{P^*} h - D_P B P^* h \\ &= D_P (A^* - B P^*) h - P^* G_2^* D_{P^*} h \\ &= D_P D_{P^*} G_1 D_{P^*} h - P^* G_2^* D_{P^*} h \\ &= (D_P D_{P^*} G_1 - P^* G_2^*) D_{P^*} h. \end{aligned}$$

The other relation can be proved similarly. ■

Observation If (N_1, N_2, N_3) is a commuting triple of bounded operators such that N_3 is unitary and $N_1 = N_2^* N_3$, then N_1 and N_2 are normal operators.

Proof. $N_1 = N_2^* N_3$ gives after multiplying N_3^* from left $N_1 N_3^* = N_2^*$, which after taking adjoint each side gives $N_2 = N_3 N_1^* = N_1^* N_3$, where last equality follows from Fuglede-Putnam's theorem (See [3]).

$$N_1 N_1^* N_3 = N_1 N_2 = N_2 N_1 = N_1^* N_3 N_1 = N_1^* N_1 N_3.$$

Since N_3 is unitary, it follows that N_1 is a normal operator. Also

$$N_2 N_2^* N_3 = N_2 N_1 = N_1 N_2 = N_2^* N_3 N_2 = N_2^* N_2 N_3.$$

Since N_3 is unitary, it follows that N_2 is normal operator. ■

3. DILATION OF A TETRABLOCK CONTRACTION - PROOF OF THEOREM 4

We begin this section by showing that the fundamental operators F_1 and F_2 of a tetrablock contraction (A, B, P) and the fundamental operators G_1 and G_2 of the adjoint tetrablock contraction (A^*, B^*, P^*) are intimately related in the sense explained in the following lemma.

Lemma 10. Let (A, B, P) be a tetrablock contraction on a Hilbert space \mathcal{H} and F_1, F_2 and G_1, G_2 be fundamental operators of (A, B, P) and (A^*, B^*, P^*) respectively. Then

$$[F_1, F_2] = 0 \text{ and } [F_1, F_1^*] = [F_2, F_2^*]$$

if and only if

$$[G_1, G_2] = 0 \text{ and } [G_1, G_1^*] = [G_2, G_2^*].$$

Proof. Note that if we can show one of the above implications, then the other will follow from it. Because once we prove one implication, we then apply it for the tetrablock contraction (A^*, B^*, P^*) to get the other implication. So we shall prove only one implication.

We first show that if V on a Hilbert space $\tilde{\mathcal{H}}$ is the minimal isometric dilation of a contraction P on Hilbert space \mathcal{H} , then dimensions of the spaces \mathcal{D}_{V^*} and \mathcal{D}_{P^*} are the same. Since V is minimal, $\tilde{\mathcal{H}} = \overline{\text{span}}\{V^n h : h \in \mathcal{H}, n \geq 0\}$. We have for all $h, h' \in \mathcal{H}$ and $n \geq 0$, $\langle V^* h, V^n h' \rangle = \langle h, V^{n+1} h' \rangle = \langle h, P^{n+1} h' \rangle = \langle P^* h, P^n h' \rangle = \langle P^* h, V^n h' \rangle$. Hence $V^*|_{\mathcal{H}} = P^*$. Note that $D_{V^*}^2 V^n h = (I - VV^*)V^n h = 0$, for all $n \geq 1$. So the operator $D_{V^*}^2$ kills $\{V^n h, h \in \mathcal{H} \text{ and } n \geq 1\}$, so does the operator D_{V^*} . Therefore $\mathcal{D}_{V^*} = \overline{D_{V^*} \tilde{\mathcal{H}}} = \overline{D_{V^*} \mathcal{H}}$. Something more is true, $\|D_{V^*} h\|^2 = \langle (I - VV^*)h, h \rangle = \|h\|^2 - \|V^* h\|^2 = \|h\|^2 - \|P^* h\|^2 = \|D_{P^*} h\|^2$. So we define a unitary $X : \mathcal{D}_{P^*} \rightarrow \mathcal{D}_{V^*}$ by $XD_{P^*} h = D_{V^*} h$, for all $h \in \mathcal{H}$ and extend it to the closure continuously. That X is a unitary, is clear from its very definition and from the fact that $\|D_{V^*} h\| = \|D_{P^*} h\|$, for all $h \in \mathcal{H}$. Note that the unitary X satisfies $XD_{P^*} = D_{V^*}$, whenever V is the minimal isometric dilation of P .

Let us suppose that the fundamental operators F_1 and F_2 of the tetrablock contraction (A, B, P) satisfy $[F_1, F_2] = 0$ and $[F_1, F_1^*] = [F_2, F_2^*]$. Then we know from [2](Theorem 6.1) that (A, B, P) has a tetrablock isometric dilation (V_1, V_2, V_3) such that V_3 is the minimal isometric dilation of P and in fact, (V_1, V_2, V_3) is a co-extension of (A, B, P) . We shall prove that $[G_1, G_2] = 0$ and $[G_1, G_1^*] = [G_2, G_2^*]$, where G_1 and G_2 are fundamental operators of (A^*, B^*, P^*) .

Now we show that XG_1X^* and XG_2X^* are the fundamental operators of the tetrablock co-isometry (V_1^*, V_2^*, V_3^*) , where X is the unitary defined above. For $h, h' \in \mathcal{H}$ we have

$$\begin{aligned} & \langle (A^* - BP^*)h, h' \rangle = \langle D_{P^*} G_1 D_{P^*} h, h' \rangle \\ \Rightarrow & \langle A^* h, h' \rangle - \langle P^* h, B^* h' \rangle = \langle G_1 D_{P^*} h, D_{P^*} h' \rangle \\ \Rightarrow & \langle V_1^* h, h' \rangle - \langle V_3^* h, V_2^* h' \rangle = \langle G_1 X^* D_{V_3^*} h, X^* D_{V_3^*} h' \rangle \\ \Rightarrow & \langle (V_1^* - V_2^* V_3^*)h, h' \rangle = \langle D_{V_3^*} X G_1 X^* D_{V_3^*} h, h' \rangle. \end{aligned}$$

Therefore $(V_1^* - V_2^* V_3^*) = D_{V_3^*} (X G_1 X^*) D_{V_3^*}$. Similarly it can be showed that $(V_2^* - V_1^* V_3^*) = D_{V_3^*} (X G_2 X^*) D_{V_3^*}$. By uniqueness of the fundamental operators, $X G_1 X^*$ and $X G_2 X^*$ are the fundamental operators of (V_1^*, V_2^*, V_3^*) .

Since (V_1, V_2, V_3) is a tetrablock isometry on the Hilbert space \mathcal{K} , the space \mathcal{K} can be decomposed into $\mathcal{K}_1 \oplus \mathcal{K}_2$ in such a way that \mathcal{K}_1 and \mathcal{K}_2 reduce (V_1, V_2, V_3) and $(V_1, V_2, V_3)|_{\mathcal{K}_1} =: (\tilde{V}_1, \tilde{V}_2, \tilde{V}_3)$ is pure tetrablock isometry and $(V_1, V_2, V_3)|_{\mathcal{K}_2}$ is tetrablock unitary. See [2](Theorem 5.6) for a proof of this fact. Note that if the fundamental operators of the pure tetrablock co-isometry $(\tilde{V}_1^*, \tilde{V}_2^*, \tilde{V}_3^*)$ are \tilde{G}_1 and \tilde{G}_2 , then the fundamental operators of the tetrablock co-isometry (V_1^*, V_2^*, V_3^*) are $0 \oplus \tilde{G}_1$ and $0 \oplus \tilde{G}_2$.

By Corollary 15 of [7], we have that the pure tetrablock isometry $(\tilde{V}_1, \tilde{V}_2, \tilde{V}_3)$ is unitarily equivalent to $(M_{\tilde{G}_1^* + \tilde{G}_2 z}, M_{\tilde{G}_2^* + \tilde{G}_1 z}, M_z)$, which acts on the Hilbert space $H^2(\mathcal{D}_{\tilde{V}_3^*})$. By commutativity of the triple $(M_{\tilde{G}_1^* + \tilde{G}_2 z}, M_{\tilde{G}_2^* + \tilde{G}_1 z}, M_z)$ we get

$$[\tilde{G}_1, \tilde{G}_2] = 0 \text{ and } [\tilde{G}_1, \tilde{G}_1^*] = [\tilde{G}_2, \tilde{G}_2^*].$$

Hence the fundamental operators of the tetrablock co-isometry (V_1^*, V_2^*, V_3^*) , which are nothing but $0 \oplus \tilde{G}_1$ and $0 \oplus \tilde{G}_2$ also satisfy the above equality. But the fundamental operators of the tetrablock co-isometry (V_1^*, V_2^*, V_3^*) , as observed above, are $X G_1 X^*$ and

XG_2X^* , where recall that X is a unitary. Hence G_1 and G_2 also satisfy the above equality. In other words G_1 and G_2 satisfy

$$[G_1, G_2] = 0 \text{ and } [G_1, G_1^*] = [G_2, G_2^*].$$

Hence the proof. ■

We now prove that (R_1, R_2, U) on \mathcal{K} defined in the statement of Theorem 4 is a tetrablock unitary. To prove that, we shall show the following

- (i) R_1, R_2 and U commute with each other,
- (ii) $R_1 = R_2^*U$ and
- (iii) R_2 is a contraction.

These will imply that (R_1, R_2, U) is a tetrablock unitary by part (2) of Theorem 5.

Proof of part (i) It can be easily checked that the operators D_1, D_2 and D_3 on $l^2(\mathcal{D}_{P^*})$ defined in Theorem 4 are unitarily equivalent to the operators $M_{G_1+G_2^*z}^*, M_{G_2+G_1^*z}^*$ and M_z^* on $H_{\mathcal{D}_{P^*}}^2(\mathbb{D})$ respectively. To show that $R_1 = \begin{pmatrix} V_1 & C_1 \\ 0 & D_1 \end{pmatrix}$ and $R_2 = \begin{pmatrix} V_2 & C_2 \\ 0 & D_2 \end{pmatrix}$ commute, we shall have to show $V_1V_2 = V_2V_1$, $D_1D_2 = D_2D_1$ and $V_1C_2 + C_1D_2 = V_2C_1 + C_2D_1$.

That V_1 and V_2 commute follows from the fact that (V_1, V_2, V_3) is a tetrablock isometry. As we have observed, the commutativity of D_1 and D_2 is equivalent to that of $M_{G_2+G_1^*z}^*$ and $M_{G_1+G_2^*z}^*$. It can be easily checked that the commutativity of $M_{G_2+G_1^*z}^*$ and $M_{G_1+G_2^*z}^*$ is equivalent to G_1, G_2 satisfying equation (1.3) in place of F_1 and F_2 respectively, which holds true. So we have $D_1D_2 = D_2D_1$. From the definition of the operators, it can be easily checked that

$$(3.1) \quad V_1C_2 + C_1D_2 = V_2C_1 + C_2D_1.$$

The detailed proof of this can be found in the Appendix.

Proof of part (ii) To prove $R_1 = R_2^*U$ it is equivalent to show the following:

$$V_1 = V_2^*V_3, \quad C_1 = V_2^*C_3, \quad C_2^*V_3 = 0 \text{ and } D_1 = C_2^*C_3 + D_2^*D_3$$

We shall check these conditions one by one. Since (V_1, V_2, V_3) is a tetrablock isometry, the first condition is satisfied.

$$\begin{aligned} V_2^*C_3(a_0, a_1, a_2, \dots) &= V_2^*(D_{P^*}a_0 \oplus (-P^*a_0, 0, 0, \dots)) \\ &= (B^*D_{P^*} - D_P F_1 P^*)a_0 \oplus (-F_2^*P^*a_0, 0, 0, \dots) \\ &= C_1(a_0, a_1, a_2, \dots) \text{ [using Lemma 8]}. \end{aligned}$$

$$\begin{aligned} C_2^*V_3(h \oplus (a_0, a_1, a_2, \dots)) &= C_2^*(Ph \oplus (D_P h, a_0, a_1, \dots)) \\ &= ((G_1^*D_{P^*}P - PF_1D_P)a_0, 0, 0, \dots) \\ &= (0, 0, 0, \dots) \text{ [using Lemma 7]}. \end{aligned}$$

$$\begin{aligned} \text{Now } (C_2^*C_3 + D_2^*D_3)(a_0, a_1, a_2, \dots) &= C_2^*(D_{P^*}a_0 \oplus (-P^*a_0, 0, 0, \dots)) + D_2^*(a_1, a_2, a_3, \dots) \\ &= ((G_1^*D_{P^*}^2 + PF_1P^*)a_0, 0, 0, \dots) + (G_2a_1, G_1^*a_1 + G_2a_2, G_1^*a_2 + G_2a_3, \dots) \\ &= ((G_1^* - G_1^*PP^* + PF_1P^*)a_0 + G_2a_1, G_1^*a_1 + G_2a_2, G_1^*a_2 + G_2a_3, \dots) \\ &= (G_1^*a_0 + G_2a_1, G_1^*a_1 + G_2a_2, G_1^*a_2 + G_2a_3, \dots) \\ &= D_1(a_0, a_1, a_2, \dots) \text{ [using Lemma 7]}. \end{aligned}$$

So this completes the proof of part (ii).

Proof of part (iii) First we shall show that $r(D_2) \leq 1$. What we actually show is that, numerical radius of D_2 is not greater than one. Since spectral radius is not greater than the numerical radius (See [6], Theorem 1.2.11.), we shall be done. Let us define

$$\varphi : \mathbb{D} \rightarrow \mathcal{B}(\mathcal{D}_{P^*}) \text{ by } \varphi(z) = G_2 + zG_1^*.$$

Clearly φ is holomorphic, bounded and continuous on the boundary $\partial\mathbb{D} = \mathbb{T}$ of the disk. As observed before in the introduction, the operator D_2^* goes to multiplication by the function φ . Now $w(M_\varphi) \leq \sup\{w(\varphi(z)) : z \in \mathbb{T}\}$. Recall that the numerical radius of an operator X is not greater than one if and only if the real part of the operator zX is not bigger than identity for every z on the unit circle, see [1]. From Theorem 3.5 of [2], we know that $w(G_1 + zG_2) \leq 1$, which in turn gives that $w(z_1G_1 + z_2G_2) \leq 1$, for all z_1, z_2 in the unit circle. Which is equivalent to $(z_1G_1 + z_2G_2) + (z_1G_1 + z_2G_2)^* \leq 2I$, which gives after rearranging $(z_2G_2 + \bar{z}_1G_1^*) + (z_2G_2 + \bar{z}_1G_1^*)^* \leq 2I$, which is equivalent to saying that $z_2(G_2 + zG_1^*) + \bar{z}_2(G_2 + zG_1^*)^* \leq 2I$, for every z and z_2 on the unit circle. Which implies that $w(G_2 + zG_1^*) \leq 1$, for all z in the unit circle. Hence $w(M_\varphi) \leq 1$.

This implies that $r(D_2) = r(M_\varphi) \leq w(M_\varphi) \leq 1$. Since R_2 of the form $\begin{pmatrix} V_2 & C_2 \\ 0 & D_2 \end{pmatrix}$, we have by Lemma 1 of [5], that $\sigma(R_2) \subseteq \sigma(V_2) \cup \sigma(D_2)$. Since $r(V_2) \leq 1$ ((V_1, V_2, V_3) being tetrablock isometry) and $r(D_2) \leq 1$, we have $r(R_2) \leq 1$. Since R_2 is a normal operator (applying the observation in previous section to the triple (R_1, R_2, U)), we have by Stampfli's theorem (which says that, if X is hyponormal, then $\|X^n\| = \|X\|^n$ and so $\|X\| = r(X)$, see Proposition 4.9 of [4]) that $\|R_2\| \leq 1$. This completes the proof of part (iii).

Therefore (R_1, R_2, U) is a tetrablock unitary. Now to prove (R_1, R_2, U) is a dilation of (A, B, P) , we observe that each of R_1, R_2 and U are upper triangular with respect to the decomposition $\tilde{\mathcal{H}} \oplus l^2(\mathcal{D}_{P^*})$ of \mathcal{K} with V_1, V_2 and V_3 in the (11)-th places of the operator matrices respectively. Also noting that each of V_1, V_2 and V_3 are lower triangular with respect to the decomposition $\mathcal{H} \oplus l^2(\mathcal{D}_P)$ of $\tilde{\mathcal{H}}$ with A, B and P in the (11)-th places of the operator matrices respectively, we get

$$A^m B^n P^l = P_{\mathcal{H}} R_1^m R_2^n U^l|_{\mathcal{H}}, \text{ for all } m, n, l \geq 0.$$

This completes the proof of Theorem 4. ■

Remark 11 (Minimality). Minimality of a commuting normal boundary dilation $\underline{N} = (N_1, N_2, \dots, N_d)$ on a space \mathcal{K} of a commuting tuple (T_1, T_2, \dots, T_d) of bounded operators on a space \mathcal{H} means that the space \mathcal{K} is no bigger than

$$\overline{\text{span}}\{N_1^{k_1} N_2^{k_2} \dots N_d^{k_d} N_1^{*l_1} N_2^{*l_2} \dots N_d^{*l_d} h : h \in \mathcal{H} \text{ where } k_i \text{ and } l_i \in \mathbb{N} \text{ for } i = 1, 2, \dots, d\}.$$

In our construction, the space is the minimal unitary dilation space of the contraction P , which is obviously a subspace of $\overline{\text{span}}\{R_1^{k_1} R_2^{k_2} R_1^{*l_1} R_2^{*l_2} U^n h : h \in \mathcal{H}, k_i, l_i \geq 0 \text{ and } n \in \mathbb{Z}\}$. Note that no dilation of (A, B, P) can take place on a space smaller than the minimal unitary dilation space of the contraction P , since the dilation has to dilate P also. Hence the dilation is minimal.

4. UNIQUENESS OF THE DILATION

In this section, we show that the minimal dilation (R_1, R_2, U) of (A, B, P) defined in Theorem 4 is unique under a suitable condition.

Note that the operator U , when written with respect to the decomposition $l^2(\mathcal{D}_P) \oplus \mathcal{H} \oplus l^2(\mathcal{D}_{P^*})$, takes the following form,

$$\begin{pmatrix} U_1 & U_2 & U_3 \\ 0 & P & U_4 \\ 0 & 0 & U_5 \end{pmatrix},$$

where U_1, U_2, U_3, U_4 and U_5 are as follows

$$\begin{aligned} U_1(a_0, a_1, a_2, \dots) &= (0, a_0, a_1, \dots), \quad U_2(h) = (D_P h, 0, 0, \dots) \\ U_3(b_0, b_1, b_2, \dots) &= (-P^* b_0, 0, 0, \dots), \quad U_4(b_0, b_1, b_2, \dots) = D_{P^*} b_0 \text{ and} \\ U_5(b_0, b_1, b_2, \dots) &= (b_1, b_2, b_3, \dots), \end{aligned}$$

for all $h \in \mathcal{H}, (a_0, a_1, a_2, \dots) \in l^2(\mathcal{D}_P)$ and $(b_0, b_1, b_2, \dots) \in l^2(\mathcal{D}_{P^*})$. Note that this is the Schäffer minimal unitary dilation of the contraction P (See [8] or ch. 1, sec. 5 in [9]).

The next result says that if $(R_1, R_2, \dots, R_{n-1}, U)$ is a dilation of $(S_1, S_2, \dots, S_{n-1}, P)$, where P is a contraction and U is the Schäffer minimal unitary dilation of P , then the operators $R_j, j = 1, 2, \dots, n-1$ can not be of arbitrary form.

Lemma 12. *Let $(R_1, R_2, \dots, R_{n-1}, U)$ on \mathcal{K} be a dilation of $(S_1, S_2, \dots, S_{n-1}, P)$ on \mathcal{H} , where P is a contraction on \mathcal{H} and U on \mathcal{K} is the Schäffer minimal unitary dilation of P . Then for all $j = 1, 2, \dots, n-1$, R_j admits a matrix representation of the form*

$$\begin{pmatrix} * & * & * \\ 0 & S_j & * \\ 0 & 0 & * \end{pmatrix},$$

with respect to the decomposition $\mathcal{K} = l^2(\mathcal{D}_P) \oplus \mathcal{H} \oplus l^2(\mathcal{D}_{P^*})$.

Proof. Let $R_j = (R_{kl}^j)_{k,l=1}^3$ with respect to $\mathcal{K} = l^2(\mathcal{D}_P) \oplus \mathcal{H} \oplus l^2(\mathcal{D}_{P^*})$ for each $j = 1, 2, \dots, n-1$. Call $\tilde{\mathcal{H}} = l^2(\mathcal{D}_P) \oplus \mathcal{H}$. Since U is minimal we have $\mathcal{K} = \bigvee_{m=-\infty}^{\infty} U^m \mathcal{H}$ and $\tilde{\mathcal{H}} = \bigvee_{m=0}^{\infty} U^m \mathcal{H} = \bigvee_{m=0}^{\infty} V^m \mathcal{H}$, where V is the minimal isometry dilation of P . Note that

$$P_{\mathcal{H}} R_j (U^m h) = S_j P^m h = S_j P_{\mathcal{H}} U^m h, \text{ for all } h \in \mathcal{H}, m \in \mathbb{N} \text{ and } j = 1, 2, \dots, n-1.$$

Hence we have $P_{\mathcal{H}} R_j|_{\tilde{\mathcal{H}}} = S_j P_{\mathcal{H}}|_{\tilde{\mathcal{H}}}$ or equivalently $S_j^* = P_{\tilde{\mathcal{H}}} R_j^*|_{\mathcal{H}}$ for all $j = 1, 2, \dots, n-1$. This shows that $R_{21}^j = 0$, for all $j = 1, 2, \dots, n-1$.

Call $\tilde{\mathcal{N}} = \mathcal{H} \oplus l^2(\mathcal{D}_{P^*})$, then note that $\tilde{\mathcal{N}} = \bigvee_{n=0}^{\infty} U^{*n} \mathcal{H}$. We have

$$P_{\mathcal{H}} R_j^* (U^{*m} h) = S_j^* P^{*m} h = S_j^* P_{\mathcal{H}} U^{*m} h, \text{ for all } h \in \mathcal{H}, m \in \mathbb{N} \text{ and } j = 1, 2, \dots, n-1.$$

This and a similar argument as above give us $S_j = P_{\tilde{\mathcal{N}}} R_j|_{\mathcal{H}}$. Therefore $R_{32}^j = 0$, for all $j = 1, 2, \dots, n-1$.

So far, we have showed that for each $j = 1, 2, \dots, n-1$, R_j admits the matrix representation of the form

$$\begin{pmatrix} R_{11}^j & R_{12}^j & R_{13}^j \\ 0 & S_j & R_{23}^j \\ R_{31}^j & 0 & R_{33}^j \end{pmatrix},$$

with respect to the decomposition $\mathcal{K} = l^2(\mathcal{D}_P) \oplus \mathcal{H} \oplus l^2(\mathcal{D}_{P^*})$. To show that $R_{13}^j = 0$ we proceed as follows. From the commutativity of R_j with U we get, by an easy matrix calculation

$$(4.1) \quad R_{31}^j U_1 = U_5 R_{31}^j \text{ and } R_{31}^j U_2 = 0,$$

(equating the 31^{th} entries and 32^{th} entries of $R_j U$ and $U R_j$ respectively). By the definition of U_2 , we have $Ran U_2 = Ran(I - U_1 U_1^*)$. Therefore $R_{31}^j(I - U_1 U_1^*) = 0$. Which with the first equation of (4.1) gives $R_{31}^j = U_5 R_{31}^j U_1^*$. Which gives after n -th iteration $R_{31}^j = U_5^n R_{31}^j U_1^{*n}$. Now since $U_1^{*n} \rightarrow 0$ as $n \rightarrow \infty$, we have that $R_{31}^j = 0$ for each $j = 1, 2, \dots, n-1$. This completes the proof of the lemma. \blacksquare

First we prove a weaker version of the uniqueness theorem, which will be used to prove the stronger version of the uniqueness theorem.

Lemma 13. *Suppose (A, B, P) is a tetrablock contraction on a Hilbert space \mathcal{H} and (R_1, R_2, U) is the above tetrablock unitary dilation of (A, B, P) . If $(\tilde{R}_1, \tilde{R}_2, U)$ is another tetrablock unitary dilation of (A, B, P) such that \tilde{R}_1 and \tilde{R}_2 are extensions of V_1 and V_2 respectively, then $\tilde{R}_1 = R_1$ and $\tilde{R}_2 = R_2$.*

Proof. Since \tilde{R}_1 and \tilde{R}_2 on the Hilbert space \mathcal{K} are such that they are extensions of V_1 and V_2 respectively, the matrix representations of \tilde{R}_1 and \tilde{R}_2 with respect to the decomposition $\mathcal{K} = \tilde{H} \oplus l^2(\mathcal{D}_{P^*})$ will be of the form $\begin{pmatrix} V_1 & \tilde{C}_1 \\ 0 & \tilde{D}_1 \end{pmatrix}$ and $\begin{pmatrix} V_2 & \tilde{C}_2 \\ 0 & \tilde{D}_2 \end{pmatrix}$ respectively, where $\tilde{C}_1, \tilde{C}_2 : l^2(\mathcal{D}_{P^*}) \rightarrow \tilde{\mathcal{H}}$ and $\tilde{D}_1, \tilde{D}_2 : l^2(\mathcal{D}_{P^*}) \rightarrow l^2(\mathcal{D}_{P^*})$ are some operators. We want to show that $\tilde{C}_1, \tilde{C}_2, \tilde{D}_1$ and \tilde{D}_2 are same as C_1, C_2, D_1 and D_2 respectively. Since $(\tilde{R}_1, \tilde{R}_2, U)$ is a tetrablock unitary, we have by part (2) of Theorem 5 the following: \tilde{R}_1, \tilde{R}_2 and unitary operator U commute, \tilde{R}_2 is a contraction, and $\tilde{R}_1 = \tilde{R}_2^* U$. The fact that U is unitary, gives us

$$(4.2) \quad D_3^* D_3 + C_3^* C_3 = I \text{ and } C_3^* V_3 = 0.$$

The fact that \tilde{R}_2 and U commute, gives us

$$(4.3) \quad V_2 C_3 + \tilde{C}_2 D_3 = V_3 \tilde{C}_2 + C_3 \tilde{D}_2 \text{ and } \tilde{D}_2 D_3 = D_3 \tilde{D}_2.$$

The fact that \tilde{R}_1 and U commute, gives us

$$(4.4) \quad V_1 C_3 + \tilde{C}_1 D_3 = V_3 \tilde{C}_1 + C_3 \tilde{D}_1 \text{ and } \tilde{D}_1 D_3 = D_3 \tilde{D}_1,$$

and commutativity of \tilde{R}_1 and \tilde{R}_2 gives

$$(4.5) \quad V_1 \tilde{C}_2 + \tilde{C}_1 \tilde{D}_2 = V_2 \tilde{C}_1 + \tilde{C}_2 \tilde{D}_1 \text{ and } \tilde{D}_1 \tilde{D}_2 = \tilde{D}_2 \tilde{D}_1.$$

Since $\tilde{R}_1 = \tilde{R}_2^* U$, we have

$$(4.6) \quad \tilde{C}_1 = V_2^* C_3 \text{ and } \tilde{D}_1 = \tilde{C}_2^* C_3 + \tilde{D}_2^* D_3.$$

Therefore

$$\begin{aligned} \tilde{C}_1(a_0, a_1, a_2, \dots) &= V_2^* C_3(a_0, a_1, a_2, \dots) \\ &= V_2^*(D_{P^*} a_0 \oplus (-P^* a_0, 0, 0, \dots)) \\ &= ((B^* D_{P^*} - D_P F_1 P^*) a_0 \oplus (-F_2^* P^* a_0, 0, 0, \dots)) \\ &= (D_{P^*} G_2 a_0 \oplus (-F_2^* P^* a_0, 0, 0, \dots)) \text{ [by Lemma 8]} \\ &= C_1(a_0, a_1, a_2, \dots). \end{aligned}$$

Also we have $\tilde{R}_2 = \tilde{R}_1^* U$, which gives

$$(4.7) \quad \tilde{C}_2 = V_1^* C_3 \text{ and } \tilde{D}_2 = \tilde{C}_1^* C_3 + \tilde{D}_1^* D_3.$$

Therefore

$$\begin{aligned}
\tilde{C}_2(a_0, a_1, a_2, \dots) &= V_1^* C_3(a_0, a_1, a_2, \dots) \\
&= V_1^*(D_{P^*}a_0 \oplus (-P^*a_0, 0, 0, \dots)) \\
&= ((A^*D_{P^*} - D_P F_2 P^*)a_0 \oplus (-F_1^* P^* a_0, 0, 0, \dots)) \\
&= (D_{P^*} G_1 a_0 \oplus (-F_1^* P^* a_0, 0, 0, \dots)) \text{ [by Lemma 8]} \\
&= C_2(a_0, a_1, a_2, \dots).
\end{aligned}$$

To find \tilde{D}_1 and \tilde{D}_2 , we proceed by multiplying (4.3) by C_3^* from left. Then using (4.2), we get

$$\tilde{D}_2^*(I - D_3^* D_3) = D_3^* \tilde{C}_2^* C_3 + C_3^* V_2^* C_3.$$

Noting that $(I - D_3^* D_3)$ is the orthogonal projection of $l^2(\mathcal{D}_{P^*})$ onto the first component, we get

$$\begin{aligned}
&\tilde{D}_2^*(a_0, 0, 0, \dots) \\
&= (D_3^* \tilde{C}_2^* C_3 + C_3^* V_2^* C_3)(a_0, a_1, a_2, \dots) \\
&= D_3^* \tilde{C}_2^*(D_{P^*}a_0 \oplus (-P^*a_0, 0, 0, \dots)) + C_3^* V_2^*(D_{P^*}a_0 \oplus (-P^*a_0, 0, 0, \dots)) \\
&= D_3^*((G_1^*(I - PP^*) + PF_1 P^*)a_0, 0, 0, \dots) \\
&\quad + C_3^*((B^*D_{P^*} - D_P F_1 P^*)a_0 \oplus (-F_2^* P^* a_0, 0, 0, \dots)) \\
&= (0, (G_1^*(I - PP^*) + PF_1 P^*)a_0, 0, 0, \dots) \\
&\quad + ((D_{P^*} B^* D_{P^*} - D_P F_1 P^* + PF_2^* P^*)a_0, 0, 0, \dots) \\
&= ((D_{P^*}(B^*D_{P^*} - D_P F_1 P^*) + PF_2^* P^*)a_0, (G_1^*(I - PP^*) + PF_1 P^*)a_0, 0, 0, \dots) \\
&= (((I - PP^*)G_2 + PF_2^* P^*)a_0, (G_1^*(I - PP^*) + PF_1 P^*)a_0, 0, 0, \dots) \text{ [by Lemma 8]} \\
&= (G_2 a_0, G_1^* a_0, 0, 0, \dots) \text{ [by Lemma 7]}
\end{aligned}$$

Therefore for $a \in \mathcal{D}_{P^*}$, we have $\tilde{D}_2^*(a, 0, 0, \dots) = (G_2 a, G_1^* a, 0, 0, \dots)$.

$$\begin{aligned}
&\tilde{D}_2^*(\overbrace{(0, \dots, 0)}^{n \text{ times}}, a, 0, \dots) \\
&= \tilde{D}_2^* D_3^{*n}(a, 0, 0, 0, \dots) \\
&= D_3^{*n} \tilde{D}_2^*(a, 0, 0, 0, \dots) \text{ [using last equation of (4.3)]} \\
&= D_3^{*n}(G_2 a, G_1^* a, 0, 0, \dots) \\
&= (\overbrace{(0, \dots, 0)}^{n \text{ times}}, G_2 a, G_1^* a, 0, 0, \dots), \text{ for every } n \geq 0..
\end{aligned}$$

Therefore for an arbitrary element $(a_0, a_1, a_2, \dots) \in l^2(\mathcal{D}_{P^*})$, we have

$$\begin{aligned}
\tilde{D}_2^*(a_0, a_1, a_2, \dots) &= \tilde{D}_2^*((a_0, 0, 0, \dots) + (0, a_1, 0, \dots) + (0, 0, a_2, \dots) + \dots) \\
&= (G_2 a_0, G_1^* a_0, 0, 0, \dots) \\
&\quad + (0, G_2 a_1, G_1^* a_1, 0, 0, \dots) + (0, 0, G_2 a_2, G_1^* a_2, 0, 0, \dots) + \dots \\
&= (G_2 a_0, G_1^* a_0 + G_2 a_1, G_1^* a_1 + G_2 a_2, \dots).
\end{aligned}$$

For (a_0, a_1, a_2, \dots) and (b_0, b_1, b_2, \dots) in $l^2(\mathcal{D}_{P^*})$, we have

$$\begin{aligned}
& \langle (a_0, a_1, a_2, \dots), \tilde{D}_2^*(b_0, b_1, b_2, \dots) \rangle \\
&= \langle (a_0, a_1, a_2, \dots), (G_2^*b_0, G_1^*b_0 + G_2b_1, G_1^*b_1 + G_2b_2, \dots) \rangle \\
&= \langle a_0, G_2b_0 \rangle + \langle a_1, G_1^*b_0 + G_2b_1 \rangle + \langle a_2, G_1^*b_1 + G_2b_2 \rangle + \dots \\
&= \langle G_2^*a_0 + G_1a_1, b_0 \rangle + \langle G_2^*a_1 + G_1a_2, b_1 \rangle + \langle G_2^*a_2 + G_1a_3, b_2 \rangle + \dots \\
&= \langle (G_2^*a_0 + G_1a_1, G_2^*a_1 + G_1a_2, G_2^*a_2 + G_1a_3, \dots), (b_0, b_1, b_2, \dots) \rangle.
\end{aligned}$$

Hence by definition of adjoint of an operator we have

$$\tilde{D}_2(a_0, a_1, a_2, \dots) = (G_2^*a_0 + G_1a_1, G_2^*a_1 + G_1a_2, G_2^*a_2 + G_1a_3, \dots) = D_2(a_0, a_1, a_2, \dots),$$

for every $(a_0, a_1, a_2, \dots) \in l^2(\mathcal{D}_{P^*})$. Therefore $\tilde{R}_2 = R_2$.

Similarly, multiplying (4.4) by C_3^* from left, using (4.2) and proceeding in the same way as above one gets \tilde{D}_1 to be

$$\tilde{D}_1(a_0, a_1, a_2, \dots) = (G_1^*a_0 + G_2a_1, G_1^*a_1 + G_2a_2, G_1^*a_2 + G_2a_3, \dots) = D_1(a_0, a_1, a_2, \dots).$$

Hence $\tilde{R}_1 = R_1$. This completes the proof. \blacksquare

Now we are ready to state and prove the stronger version of the uniqueness theorem, the main result of this section.

Theorem 14 (Uniqueness). *Let (A, B, P) be a tetrablock contraction on a Hilbert space \mathcal{H} and (R_1, R_2, U) as defined in Theorem 4, be the tetrablock unitary dilation of (A, B, P) .*

- (i) *If $(\tilde{R}_1, \tilde{R}_2, U)$ is another tetrablock unitary dilation of (A, B, P) , then $\tilde{R}_1 = R_1$ and $\tilde{R}_2 = R_2$.*
- (ii) *If $(\tilde{R}_1, \tilde{R}_2, \tilde{U})$ on some Hilbert space $\tilde{\mathcal{K}}$ containing \mathcal{H} , is another tetrablock unitary dilation of (A, B, P) where \tilde{U} is a minimal unitary dilation of P , then $(\tilde{R}_1, \tilde{R}_2, \tilde{U})$ is unitarily equivalent to (R_1, R_2, U) .*

Proof. (i) Since $(\tilde{R}_1, \tilde{R}_2, U)$ is a dilation of (A, B, P) , by Lemma 12 \tilde{R}_1 and \tilde{R}_2 are of the form

$$\begin{pmatrix} T_1 & \tilde{R}'_{12} \\ 0 & \tilde{R}'_{22} \end{pmatrix} \text{ and } \begin{pmatrix} T_2 & \tilde{R}''_{12} \\ 0 & \tilde{R}''_{22} \end{pmatrix} \text{ respectively,}$$

with respect to the decomposition $\tilde{\mathcal{H}} \oplus l^2(\mathcal{D}_{P^*})$. Where T_1 and T_2 are operators on $\tilde{\mathcal{H}}$, which admit the matrix representation

$$\begin{pmatrix} T'_{11} & T'_{12} \\ 0 & A \end{pmatrix} \text{ and } \begin{pmatrix} T''_{11} & T''_{12} \\ 0 & B \end{pmatrix} \text{ respectively,}$$

with respect to the decomposition $l^2(\mathcal{D}_P) \oplus \mathcal{H}$. Since (T_1, T_2, V) on $\tilde{\mathcal{H}}$ is the restriction of the tetrablock contraction $(\tilde{R}_1, \tilde{R}_2, U)$ to $\tilde{\mathcal{H}}$ and V is an isometry, we have (T_1, T_2, V) a tetrablock isometry. Also note that $T_1^*|_{\mathcal{H}} = A^*$, $T_2^*|_{\mathcal{H}} = B^*$ and $V^*|_{\mathcal{H}} = P^*$. So (T_1, T_2, V) is a tetrablock isometric dilation of (A, B, P) , where V is the Schäffer minimal isometric dilation of P . Now it follows from the argument given in the proof of Theorem 6.1(2) of [2] that $T_1 = V_1$ and $T_2 = V_2$, where V_1 and V_2 are as in Theorem 4. Therefore \tilde{R}_1 and \tilde{R}_2 are extensions of V_1 and V_2 respectively. Now the proof follows from Lemma 13.

(ii) Since \tilde{U} is a minimal unitary dilation of P , there exists a unitary operator $W : \tilde{\mathcal{K}} \rightarrow \mathcal{K}$ such that $W\tilde{U}W^* = U$ and $Wh = h$ for all $h \in \mathcal{H}$. This shows that $(W\tilde{R}_1W^*, W\tilde{R}_2W^*, W\tilde{U}W^*)$ is another tetrablock unitary dilation of (A, B, P) . But

$W\tilde{U}W^* = U$. Hence by part (i) we have $(W\tilde{R}_1W^*, W\tilde{R}_2W^*, W\tilde{U}W^*) = (R_1, R_2, U)$. This completes the proof of part (ii). \blacksquare

We conclude with the following remark.

Remark 15. The tetrablock unitary dilation we have constructed in this section, is simple because it acts on a small space, viz., the minimal unitary dilation space of the contraction P . Moreover, we have showed that if the unitary part of the tetrablock unitary dilation is the minimal unitary dilation of the contraction P , then the tetrablock unitary dilation is unique upto unitary equivalence.

5. APPENDIX

Here we prove that $V_1C_2 + C_1D_2 = V_2C_1 + C_2D_1$, as in equation (3.1).

From the definition of the operators V_1, V_2 and V_3 on $\tilde{\mathcal{H}}$, it is easy to see that they have the following matrix forms with respect to the decomposition $\mathcal{H} \oplus \mathcal{D}_{P^*} \oplus \mathcal{D}_{P^*} \oplus \mathcal{D}_{P^*} \oplus \dots$

$$\begin{pmatrix} A & 0 & 0 & 0 & \dots \\ F_2^*D_P & F_1 & 0 & 0 & \dots \\ 0 & F_2^* & F_1 & 0 & \dots \\ 0 & 0 & F_2^* & F_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \begin{pmatrix} B & 0 & 0 & 0 & \dots \\ F_1^*D_P & F_2 & 0 & 0 & \dots \\ 0 & F_1^* & F_2 & 0 & \dots \\ 0 & 0 & F_1^* & F_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } \begin{pmatrix} P & 0 & 0 & 0 & \dots \\ D_P & 0 & 0 & 0 & \dots \\ 0 & I & 0 & 0 & \dots \\ 0 & 0 & I & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

respectively. The operators C_1, C_2 and $C_3 : l^2(\mathcal{D}_{P^*}) \rightarrow \tilde{\mathcal{H}}$ are of the form

$$\begin{pmatrix} D_{P^*}G_2 & 0 & 0 & 0 & \dots \\ -F_2^*P^* & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \begin{pmatrix} D_{P^*}G_1 & 0 & 0 & 0 & \dots \\ -F_1^*P^* & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } \begin{pmatrix} D_{P^*} & 0 & 0 & 0 & \dots \\ -P^* & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

respectively. Finally the operators D_1, D_2 and D_3 on $l^2(\mathcal{D}_{P^*})$ are of the form

$$\begin{pmatrix} G_1^* & G_2 & 0 & 0 & \dots \\ 0 & G_1^* & G_2 & 0 & \dots \\ 0 & 0 & G_1^* & G_2 & \dots \\ 0 & 0 & 0 & G_1^* & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \begin{pmatrix} G_2^* & G_1 & 0 & 0 & \dots \\ 0 & G_2^* & G_1 & 0 & \dots \\ 0 & 0 & G_2^* & G_1 & \dots \\ 0 & 0 & 0 & G_2^* & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } \begin{pmatrix} 0 & I & 0 & 0 & \dots \\ 0 & 0 & I & 0 & \dots \\ 0 & 0 & 0 & I & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

respectively.

$$\begin{aligned} & (V_1C_2 + C_1D_2)(a_0, a_1, a_2, \dots) \\ &= V_1(D_{P^*}G_1a_0 \oplus (-F_1^*P^*a_0, 0, 0, \dots)) + C_1(G_2^*a_0 + G_1a_1, G_2^*a_1 + G_1a_2, G_2^*a_2 + G_1a_3, \dots) \\ &= (AD_{P^*}G_1a_0 \oplus ((F_2^*D_P D_{P^*}G_1 - F_1F_1^*P^*)a_0, -F_2^*F_1^*P^*a_0, 0, 0, \dots)) + \\ & \quad ((D_{P^*}G_2G_2^*a_0 + D_{P^*}G_2G_1a_1) \oplus (-F_2^*P^*G_2^*a_0 - F_2^*P^*G_1a_1, 0, 0, \dots)) \\ &= ((AD_{P^*}G_1 + D_{P^*}G_2G_2^*)a_0 + D_{P^*}G_2G_1a_1) \oplus \\ & \quad ((F_2^*D_P D_{P^*}G_1 - F_1F_1^*P^* - F_2^*P^*G_2^*)a_0 - F_2^*P^*G_1a_1, -F_2^*F_1^*P^*a_0, 0, 0, \dots) \end{aligned}$$

and

$$\begin{aligned}
& (V_2C_1 + C_2D_1)(a_0, a_1, a_2, \dots) \\
&= V_2(D_{P^*}G_2a_0 \oplus (-F_2^*P^*a_0, 0, 0, \dots)) + C_2(G_1^*a_0 + G_2a_1, G_1^*a_1 + G_2a_2, G_1^*a_2 + G_2a_3, \dots) \\
&= (BD_{P^*}G_2a_0 \oplus ((F_1^*D_{P^*}D_{P^*}G_2 - F_2F_2^*P^*)a_0, -F_1^*F_2^*P^*a_0, 0, 0, \dots)) + \\
&\quad ((D_{P^*}G_1G_1^*a_0 + D_{P^*}G_1G_2a_1) \oplus (-F_1^*P^*G_1^*a_0 - F_1^*P^*G_2a_1, 0, 0, \dots)) \\
&= ((BD_{P^*}G_2 + D_{P^*}G_1G_1^*)a_0 + D_{P^*}G_1G_2a_1) \oplus \\
&\quad ((F_1^*D_{P^*}D_{P^*}G_2 - F_2F_2^*P^* - F_1^*P^*G_1^*)a_0 - F_1^*P^*G_2a_1, -F_1^*F_2^*P^*a_0, 0, 0, \dots).
\end{aligned}$$

Hence $V_1C_2 + C_1D_2 = V_2C_1 + C_2D_1$ holds if and only if the following holds:

- (a) $AD_{P^*}G_1 + D_{P^*}G_2G_1^* = BD_{P^*}G_2 + D_{P^*}G_1G_1^*$, $D_{P^*}G_2G_1 = D_{P^*}G_1G_2$,
- (b) $F_2^*D_{P^*}D_{P^*}G_1 - F_1F_1^*P^* - F_2^*P^*G_2^* = F_1^*D_{P^*}D_{P^*}G_2 - F_2F_2^*P^* - F_1^*P^*G_1^*$, $F_2^*P^*G_1 = F_1^*P^*G_2$, and
- (c) $F_2^*F_1^*P^* = F_1^*F_2^*P^*$.

Using Lemma 6 we get that (a) holds if and only if

$$(G_1D_{P^*} + G_2^*D_{P^*}P^*)^*G_1 + D_{P^*}G_2G_2^* = (G_2D_{P^*} + G_1^*D_{P^*}P^*)^*G_2 + D_{P^*}G_1G_1^*$$

holds. After rearranging this equation we get

$$D_{P^*}G_1^*G_1 + PD_{P^*}G_2G_1 + D_{P^*}G_2G_2^* = D_{P^*}G_2^*G_2 + PD_{P^*}G_1G_2 + D_{P^*}G_1G_1^*$$

which is equivalent to saying that $D_{P^*}[G_1, G_1^*] + PD_{P^*}G_1G_2 = D_{P^*}[G_2, G_2^*] + PD_{P^*}G_2G_1$, which is true since G_1, G_2 satisfy equation (1.3) in place of F_1 and F_2 respectively. Hence (a) holds.

Note that the first part of equation (b) is equivalent to

$$\begin{aligned}
& F_2^*D_{P^*}D_{P^*}G_1 + F_2F_2^*P^* - F_2^*P^*G_2^* = F_1^*D_{P^*}D_{P^*}G_2 + F_1F_1^*P^* - F_1^*P^*G_1^* \\
&\Leftrightarrow F_2^*(D_{P^*}D_{P^*}G_1 - P^*G_2^*) + F_2F_2^*P^* = F_1^*(D_{P^*}D_{P^*}G_2 - P^*G_1^*) + F_1F_1^*P^* \\
&\Leftrightarrow F_2^*(F_1^*D_{P^*}D_{P^*} - F_2P^*) + F_2F_2^*P^* = F_1^*(F_2^*D_{P^*}D_{P^*} - F_1P^*) + F_1F_1^*P^* \text{ [by Lemma 9]} \\
&\Leftrightarrow F_2^*F_1^*D_{P^*}D_{P^*} - F_2^*F_2P^* + F_2F_2^*P^* = F_1^*F_2^*D_{P^*}D_{P^*} - F_1^*F_1P^* + F_1F_1^*P^* \\
&\Leftrightarrow [F_2^*, F_1^*]D_{P^*}D_{P^*} + [F_2, F_2^*]P^* = [F_1, F_1^*]P^*,
\end{aligned}$$

which is true by (1.3). The second part of (b) follows from Lemma 7 and from (1.3). Hence (b) holds.

Equation (c) is simply a consequence of (1.3). Hence $V_1C_2 + C_1D_2 = V_2C_1 + C_2D_1$. Consequently R_1 and R_2 commute.

Now let us show that $R_1U = UR_1$. From easy matrix calculations we get that R_1 and U commute if and only if the following holds:

$$V_1V_3 = V_3V_1, D_1D_3 = D_3D_1 \text{ and } V_1C_3 + C_1D_3 = V_3C_1 + C_3D_1.$$

The first condition holds because (V_1, V_2, V_3) is a tetrablock contraction. The operators D_1 and D_3 commute because their copy $M_{G_1+G_2^*z}$ and M_z on $H_{\mathcal{D}_{P^*}}^2(\mathbb{D})$ do. Now

$$\begin{aligned}
& V_1C_3 + C_1D_3(a_0, a_1, a_2, \dots) \\
&= V_1(D_{P^*}a_0 \oplus (-P^*a_0, 0, 0, \dots)) + C_1(a_1, a_2, a_3, \dots) \\
&= (AD_{P^*}a_0 \oplus ((F_2^*D_{P^*}D_{P^*} - F_1P^*)a_0, -F_2^*P^*a_0, 0, 0, \dots)) \\
&\quad + (D_{P^*}G_2a_1 \oplus (-F_2^*P^*a_1, 0, 0, \dots)) \\
&= (AD_{P^*}a_0 + D_{P^*}G_2a_1) \oplus ((F_2^*D_{P^*}D_{P^*} - F_1P^*)a_0 - F_2^*P^*a_1, -F_2^*P^*a_0, 0, 0, \dots)
\end{aligned}$$

and

$$\begin{aligned}
& (V_3C_1 + C_3D_1)(a_0, a_1, a_2, \dots) \\
&= V_3(D_{P^*}G_2a_0 \oplus (-F_2^*P^*a_0, 0, 0, \dots)) + C_3(G_1^*a_0 + G_2a_1, G_1^*a_1 + G_2a_2, G_1^*a_2 + G_2a_3, \dots) \\
&= (PD_{P^*}G_2a_0 \oplus (D_P D_{P^*}G_2a_0, -F_2^*P^*a_0, 0, 0, \dots)) \\
&\quad + ((D_{P^*}G_1^*a_0 + D_{P^*}G_2a_1) \oplus (-P^*G_1^*a_0 - P^*G_2a_1, 0, 0, \dots)) \\
&= ((PD_{P^*}G_2 + D_{P^*}G_1^*)a_0 + D_{P^*}G_2a_1) \\
&\quad \oplus ((D_P D_{P^*}G_2 - P^*G_1^*)a_0 - P^*G_2a_1, -F_2^*P^*a_0, 0, 0, \dots).
\end{aligned}$$

Note that the equality of $V_1C_3 + C_1D_3$ and $V_3C_1 + C_3D_1$ follows trivially from Lemma 6, Lemma 9 and Lemma 7. The commutativity of R_2 and U can be proved similarly, we skip this. This completes the proof of $V_1C_2 + C_1D_2 = V_2C_1 + C_2D_1$.

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